HINT ON 12.5.7

$$x^2y'' + xy' + \lambda y = 0$$

This is a special kind of ODE called Cauchy-Euler equation. This was taught in Math 240, but we give a hint on how to solve it in case someone did not take that course. If you consider the change of variable $x = e^t$, the chain rule tells you $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = e^t\frac{dy}{dx} = x\frac{dy}{dx}$. Moreover,

$$\frac{d^2y}{dt^2} = \frac{d}{dt}(\frac{dy}{dt}) = \frac{d}{dt}(x\frac{dy}{dx}) = \frac{dx}{dt}\frac{d}{dx}(x\frac{dy}{dx}) = e^t[\frac{dy}{dx} + x\frac{d^2y}{dx^2}] = x\frac{dy}{dx} + x^2\frac{dy^2}{dx^2}$$

Therefore the first two terms $x^2y'' + xy'$ is just $\frac{d^2y}{dt^2}$. Therefore the equation becomes $\frac{d^2y}{dt^2} + \lambda y = 0$. The boundary conditions should also be changed, since when $x = 1$ this means $t = 0$, when $x = 5$ this means $t = \ln 5$. Therefore the Sturm-Liouville problem becomes $\frac{d^2y}{dt^2} + \lambda y = 0, y(0) = 0, y(\ln 5) = 0$. This reduces to the regular Sturm-Liouville problem, and do not forget to transform everything back to x.